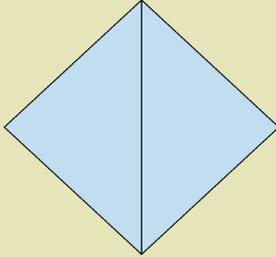




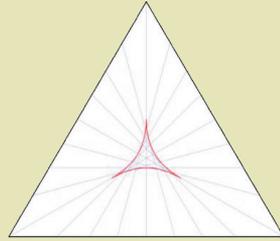
Julia Robinson
(1919 - 1985)

Halving

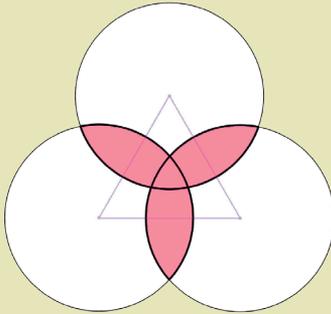
by Ken Fan, Founder Girls' Angle www.GirlsAngle.org



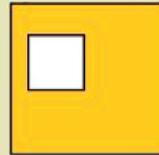
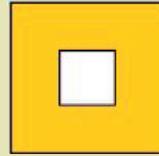
Halving a Square



Bisection Envelope



Zindler's Flower



Squares with Holes

"The JRMF really gets it right. Usually the best parts of mathematics are kept away from the public, as if you needed to be a mathematician to get to the fun stuff! It's refreshing to see a festival that brings this stuff to light, and in such a relaxed atmosphere. If you're lucky enough to have a JRMF near you, don't miss it! It's the best math party around."

– Vi Hart, Mathemusician, [youtube.com/user/ViHart](https://www.youtube.com/user/ViHart)

Julia Robinson Mathematics Festival activities are designed to open doors to higher mathematics for students in grades K-12.

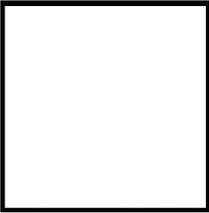
Visit www.JRMF.org for more information about our Festivals.



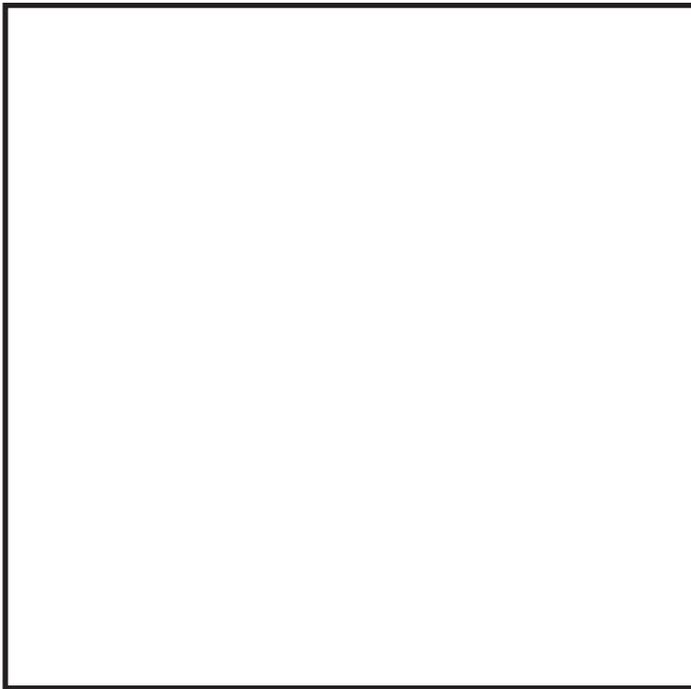
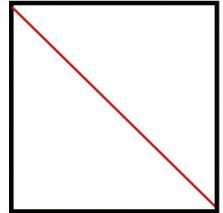
Halving!

by C. Kenneth Fan, Girls' Angle

How can you divide a shape in half?



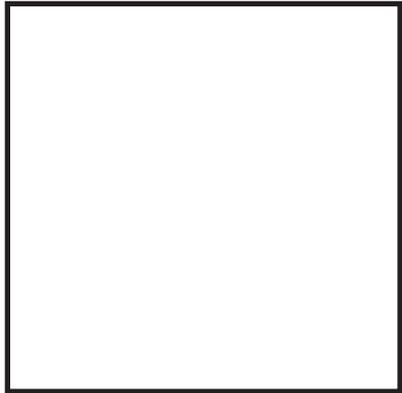
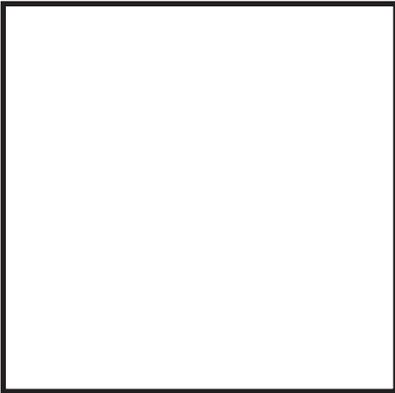
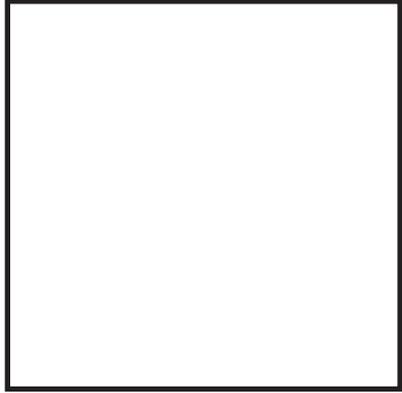
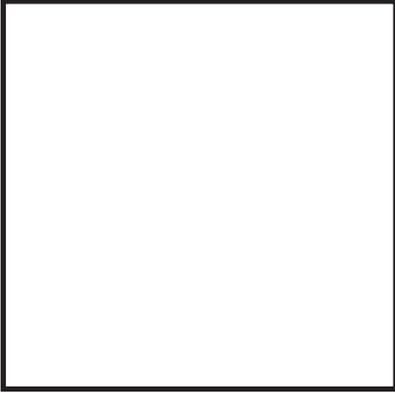
For a square, like the one on the left, you can divide it in half by slicing along a diagonal, as shown on the right. But there are many, many other ways. What's the craziest way you can think of to halve a square? Experiment with scratch paper, then carefully transfer your favorite method onto the square below.



My Halving

Make sure that you can convince yourself and others that you have *exactly* halved the square.

Four More Halvings!



You might have had more than one idea for how to halve a square. If you didn't, try hard to think of more ways. Can you come up with at least four more ways to halve a square that are so distinct from each other that you would use radically different arguments to convince people that the square has been halved? If you're really stuck, brainstorm with friends.

Create a gallery of everybody's halvings. Discuss them. Classify them according to the methods used to create them and the arguments given to prove that each square is truly halved. Make a list of the mathematical concepts that arose in your discussions.

Halving Rules!

With so many possibilities, a fun, challenging, fascinating opportunity opens up to us: Create “halving rules” that limit the possibilities and explore the effect of *your rules*.

What halving rules can you come up with? Which do you find most interesting? Write down your favorite halving rules in the box below.

My Favorite Halving Rules

Since I can't predict the rules you'll come up with, I'll proceed with the following rule:

The shape must be split in half with a single line.

However, I urge you to explore the consequences of your own rules. If you're not sure how to proceed, you can mirror what we are about to do with the one-line halving rule with *your* rules.

Let's revisit the square and analyze this one-line halving rule.

Find every way of halving a square using a single line.
Do the halving lines have anything in common with each other?

In fact, given any shape, there will always be precisely one horizontal line that splits it in half. And since no particular orientation of the shape was specified, we can find a halving line in *any* particular direction by rotating as necessary and finding the horizontal line that splits the rotated shape.

← Can you
prove this
fact?

Design Your Own Shape!

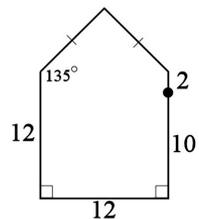
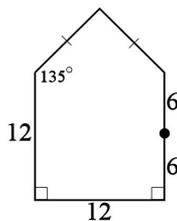
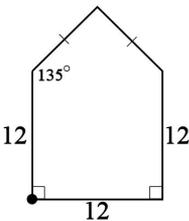
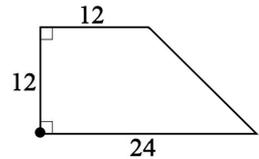
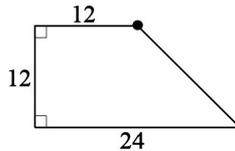
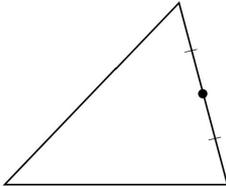
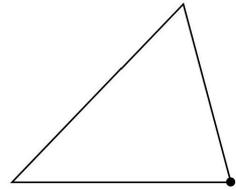
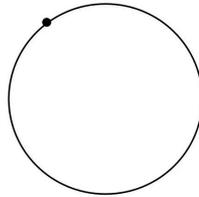
If you find the square rather, well, square, now's your chance to explore different shapes!

On scratch paper, design a shape that you particularly like. You must be able to specify the shape so precisely that you could get a friend to replicate it exactly by using only verbal instructions (such as if you were speaking to them over the phone). When you're happy with your shape, draw it below.

Now find the halving lines of your shape!

Typically, not all halving lines are equally apparent. If you were unable to find all the halving lines for your shape, can you articulate what it is about your shape that made finding the ones you were able to find possible? Then try to design a shape that you think is both interesting *and* for which you can find *all* the halving lines. When you're happy with your shape, copy it into the space below. Write or give a presentation in which you explain your findings. Perhaps include how you explored and came up with your findings.

With so many halving lines here's another challenge: Devise interesting ways to single out one from many. For example, below are a series of shapes each with a dot on its boundary. For each, find the one line that divides the shape in half and passes through the dot. Be precise! (It might help to review formulas for the areas of basic shapes.)



Reprinted from the Girls' Angle Bulletin, Volume 3, Number 2, page 15.

Invent more problems like these.

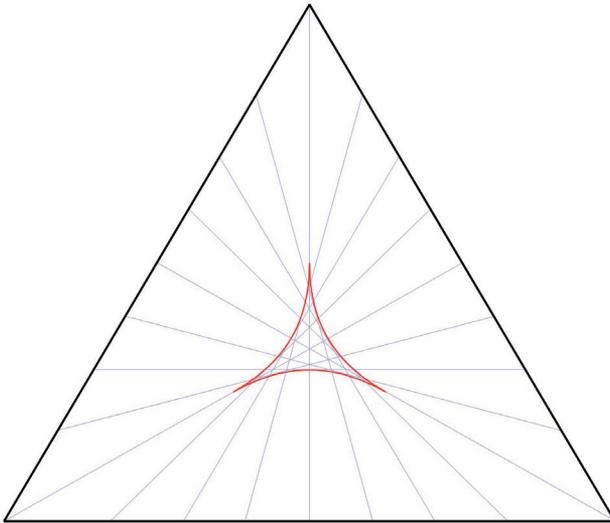
More to contemplate:

What can you say about a figure whose halving lines are all concurrent (that is, they all pass through the same point)?

Where would you draw the three halving lines that are parallel to the sides of an equilateral triangle? Are they concurrent?

Determine all the halving lines of an equilateral triangle.

The last exercise leads to some interesting mathematics. It so happens that there is a special curve inside an equilateral triangle with the property that any line tangent to the curve is a halving line, and, conversely, every halving line is tangent to the curve. This special curve, shown in red in the figure below, has been dubbed the "bisection envelope" by Noah Fechter-Pradines.¹



The gray lines are halving lines. Each one is tangent to the red curve, which is called the bisection envelope.

Noah discovered a number of interesting properties of the bisection envelope. For example, through points outside the bisection envelope, there is a unique halving line. Inside the bisection envelope (in the case of an equilateral triangle), each point is on exactly three halving lines.

Sketch the bisection envelope of a regular pentagon as accurately as you can.

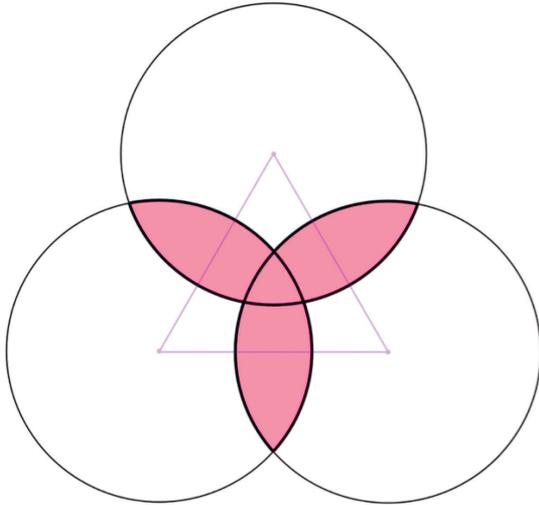
For a regular pentagon, how many halving lines are there that pass through a given point? How does this number relate to the different regions that the pentagon is partitioned into by the bisection envelope?

Sketch the bisection envelope of a semicircle as accurately as you can.

¹ See "Bisection Envelopes," *Involve*, Volume 8, Number 2 (2015), pp.307-328.

Zindler's Flower

The shaded shape shown below has rather special halving lines.



The shape is formed by the overlapping of three congruent circles centered at each of the three vertices of an equilateral triangle. The radii of the circles are about $\frac{2}{3}$ the side length of the equilateral triangle. The shape is called Zindler's flower after the Austrian mathematician Konrad Zindler (1866-1934).

What symmetries does Zindler's flower enjoy?

Do these symmetries help you to discover any of its halving lines?

We know that there are halving lines in every direction. Find them all.

What do the halving lines of Zindler's flower have in common?

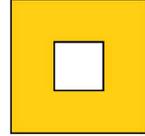
What other shapes can you think of whose halving lines enjoy the same type of commonality enjoyed by those of the halving lines of Zindler's flower?

What is the bisection envelope of Zindler's flower?

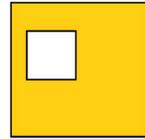
We conclude with a few miscellaneous problems.

What happens if a shape has a hole in it?

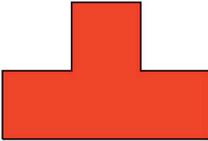
For example, suppose we remove a square from the inside of a larger square, as in the top figure. What are the halving lines of the resulting shape?



Must a halving line for the shape with a hole also halve the hole itself? What happens if the square hole isn't centered, as in the lower figure? What are the halving lines? (You can actually get infinitely many problems by changing the size and location of the hole.)

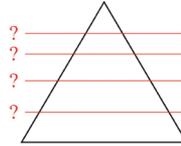


The figure is composed of four identical squares, joined edge-to-edge.

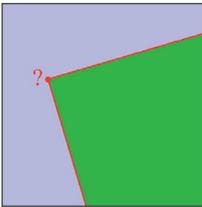


Find its halving lines.

Divide an equilateral triangle into n equal-area pieces by using lines that are all parallel to one of its sides.



Where would you place the lines?



Consider the right-angle halving rule: A shape must be halved by a right angle where the vertex of the angle falls in or on the boundary of the shape. (It's like cutting something with a knife whose blade has a 90° kink in it.)

Find all the points in a square that can serve as the vertex of a right-angled halving slice.



Please share your thoughts!

Discover any interesting halving rules and their consequences? Think of any interesting new directions or halving problems to explore? Please tell us. Send any thoughts or suggestions to the Julia Robinson Mathematics Festival (share@jrmf.org).

A crazy halving. How do you think I know half the area is yellow and half is purple?



About Girls' Angle

The problems in this booklet were proposed by Ken Fan, founder of Girls' Angle.

Girls' Angle is...

- a math club for girls;
- a comprehensive approach to math education for girls;
- a producer of math educational content;
- a supportive community for women and girls who study, use, and create mathematics;
- a nonprofit founded in 2007.

Our Mission

To foster and nurture girls' interest in mathematics and empower girls to be able to tackle any field no matter the level of mathematical sophistication.

Math Collaborations

A math collaboration is a mathematically-intense alternative to the math competition, but unlike math competitions, math collaborations are fully cooperative rather than competitive. Participants win or lose together as a unit and have every incentive to share their ideas. They must communicate well to succeed. By their nature, math collaborations can have deeper content than that of the typical math competition.

Math collaborations can be arranged for a variety of settings and any location. We have already organized long-distance events, such as one that took place at the Fitzjohn Primary School in London, England.

Contact girlsangle@gmail.com if you would like to discuss the possibility of having us host a math collaboration at your school.

About Julia Robinson Math Festival

Julia Robinson Mathematics Festival, as an organization, supports events that inspire K-12 students to explore the richness and beauty of mathematics through collaborative, creative problem-solving. We have hosted events across the United States and in over half a dozen other countries.

A Julia Robinson Mathematics Festival offers students advanced and thought-provoking mathematics in a social and cooperative atmosphere. It's an event at which students play with mathematics.

Typically, there are a dozen or more tables, each with a facilitator and a problem set, game, puzzle, or activity with mathematical themes. Students play and explore individually or in groups, share insights, and make discoveries. Facilitators provide encouragement and support, striving to ask questions rather than providing suggestions or answers.

Success is not measured by the number of problems solved nor students' speed, but rather by how long students stay engaged and persevere with activities, and by the breadth and depth of their explorations and insights.

Festivals are run locally and supported by a national network. They can address any level of student, from those struggling with mathematics to those who enjoy tackling challenging problems.

Festival activities are designed to open doors to higher mathematics for K-12 students—doors that are not at the top of the staircase, but right at street level.

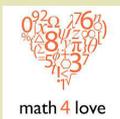
For more information, visit www.JRMF.org.

For more mathematical puzzles, visit...



NRICH promotes the learning of mathematics through problem solving. NRICH provides engaging problems, linked to the curriculum, with support for teachers. (Grades K-12)
rich.maths.org

Cool math problems that are beautiful and thought provoking. Favorite lessons and complex problems. (Grades K-6)
mathforlove.com/lesson-plan/



Julia Robinson Mathematics Festival

Explore the richness and beauty of mathematics through puzzles and problems that encourage collaborative and creative problem-solving. (Grades K-12)
jrmf.org



Wild Maths is mathematics without bounds. Visitors are free to roam and develop as mathematicians. (Grades K-12)
wild.maths.org

While a standard textbook cannot adapt to each individual learner, expii.com was created to do just that. (Grades 5-12)
expii.com and expii.com/solve

expii



Galileo.org strives to inspire a passion for learning. (Grades K-12)
galileo.org/classroom-examples/math/math-fair-problems

galileo.org



the guardian

Alex Bellos' Monday Puzzle. (Grades 5-Adult)

www.theguardian.com/science/series/alex-bellos-monday-puzzle



Gord Hamilton has a passion for getting students to realize that mathematics is beautiful. (Grades K-12)
MathPickle.com



On the NY Times website, Numberplay generally presents mathematical and/or logical puzzles and problems. (Grades 5-Adult)
nytimes.com/column/Numberplay



BRILLIANT

Brilliant's problems are created by people all over the world. Members learn how to solve problems by engaging in a vibrant community. (Grades 2-Adult)
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G4G features puzzles, games, magic tricks, and crafts. (Grades K-Adult)
www.celebrationofmind.org/puzzles-games

Math Central is an award-winning website with investigations for teachers and students. (Grades 7-12)
mathcentral.uregina.ca/mp



Dan Meyer has created problems and videos to inspire students to solve problems. (Grades 4-12)
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Gathering 4 Gardner

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Youcubed's main goal is to inspire, educate, and empower teachers of mathematics, by providing accessible and practical materials. (Grades K-12)
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A resource for educators passionate about improving students' mathematics learning and performance. (Grades K-12)
insidemathematics.org

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